$\boxed{\S 3}$

In the notation of $\boxed{\text{Thm A}}$

$$\Pi_{n^{\Box}}^{\Box} := \Pi_{X_{n^{\Box}}^{\Box}}$$

Suppose that

$$\overset{\exists}{\text{isom}}\ \Pi_{n^{\circ}}^{\circ} \xrightarrow{\sim} \Pi_{n^{\bullet}}^{\bullet}$$

By prop 3, we have

$$n^{\circ} = n^{\bullet} =: n$$

In the following, suppose that

$$n \geqslant 2.$$

$\Longrightarrow$ we want to show that

$$\underbrace{(g^{\circ}, r^{\circ}) = (g^{\bullet}, r^{\bullet})}$$

uppose that

.

show that

$(g^{\bullet}, r^{\bullet})$

Def

(i) We shall refer to

$$\mathrm{Ker}(\Pi_n \twoheadrightarrow \Pi_{n'})$$

$$\begin{tikzcd} & \pi_1 X_2 \arrow[dl, "pr_1"'] \arrow[dr, "\pi pr_2"] & \\ \pi X & & X \end{tikzcd}$$

induced by a projection $X_n \to X_{n'}$

as a *fiber subgp* of co-length n'

(or length $n-n'$)

(ii) $\alpha : \Pi_n^{\circ} \xrightarrow{\sim} \Pi_n^{\bullet}$

isom of prof gps

α : PF-admissible

$\overset{\text{def}}{\Longleftrightarrow}$ α induces a bijection between the set of fiber subgps $\subseteq \Pi_n^{\circ}$ and the set of fiber subgps $\subseteq \Pi_n^{\bullet}$.

prop4

$\alpha : \Pi_n^{\circ} \xrightarrow{\sim} \Pi_n^{\bullet}$

isom of prof gps

Suppose that

$(g^{\circ}, r^{\circ}) \notin \{(0,3),(1,1)\}$.

Then α : PF-admissible

Rem

In particular, if $(g^{\circ}, r^{\circ}) \notin \{(0,3),(1,1)\}$,

then α induces

$$\begin{tikzcd}
\Pi_n^{\circ} \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Pi_n^{\bullet} \arrow[d] \\
\Pi_2^{\circ} \arrow[r, "\sim", "\text{PF-adm}"'] & \Pi_2^{\bullet}
\end{tikzcd}$$

It s

lar, if $(g^\circ, r^\circ) \notin \{(0,3),(1,1)\}$,

$$\begin{tikzcd}
\Pi_n^\circ \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \Pi_n^\bullet \arrow[d] \\
\Pi_2^\circ \arrow[r, "\sim", "\text{PF-adm}"'] & \Pi_2^\bullet
\end{tikzcd}$$

☺

It suffices to show that

$$\underbrace{\left\{\begin{array}{c}\text{fib subgp} \subseteq \Pi_n^\circ \\ \text{of co-length } 1\end{array}\right\}}_{\substack{\| \\ \mathrm{Ker}(\Pi_n^\circ \twoheadrightarrow \Pi_1^\circ)}} \xrightarrow[\text{bij}]{\sim} \left\{\begin{array}{c}\text{fib subgp} \subseteq \Pi_n^\bullet \\ \text{of co-length } 1\end{array}\right\}$$

α induces

claim

$K^\circ \subseteq \Pi_n^\circ$: a fib subgp of co-length 1

Then $\exists K^\bullet \subseteq \Pi_n^\bullet$: //

s.t. $K^\bullet \subseteq \alpha(K^\circ)$

pf of claim

Let $\phi : \Pi_m^{\circ} \underset{\uparrow \alpha^{-1}}{\xrightarrow{\sim}} \Pi_n^{\circ} \twoheadrightarrow G := \Pi_n^{\circ}/K^{\circ}$

It suffices to show that

$\exists \underset{\uparrow}{K^{\circ}} \leq \Pi_n^{\circ}$ s.t. $\phi(K^{\circ}) = \{1\}$

fiber subgp

of co-length 1

To verify

it suffices to show that

there do not exist two

distinct fib subgps $J_1, J_2 \leq \Pi_n^{\circ}$

of length 1

s.t. $\phi(J_1) \neq \{1\}$, $\phi(J_2) \neq \{1\}$

$$K^{\circ} := \langle J \rangle_{J\,:\ \text{fib subgps of length } 1 \text{ s.t. } \phi(J) = \{1\}}$$

uffices to show that

t exist

fib subgps

$J_1, J_2 \subseteq \Pi_n$

of length 1

$1\}$, $\phi(J_2) \neq \{1\}$

$>$ J : fib subgps of length $= 1$ s.t. $\phi(J) = \{1\}$

Suppose that such J_1, J_2 exist

Then $\phi(J_1), \phi(J_2)$: non-trivial top fin gen closed normal $\subseteq G$

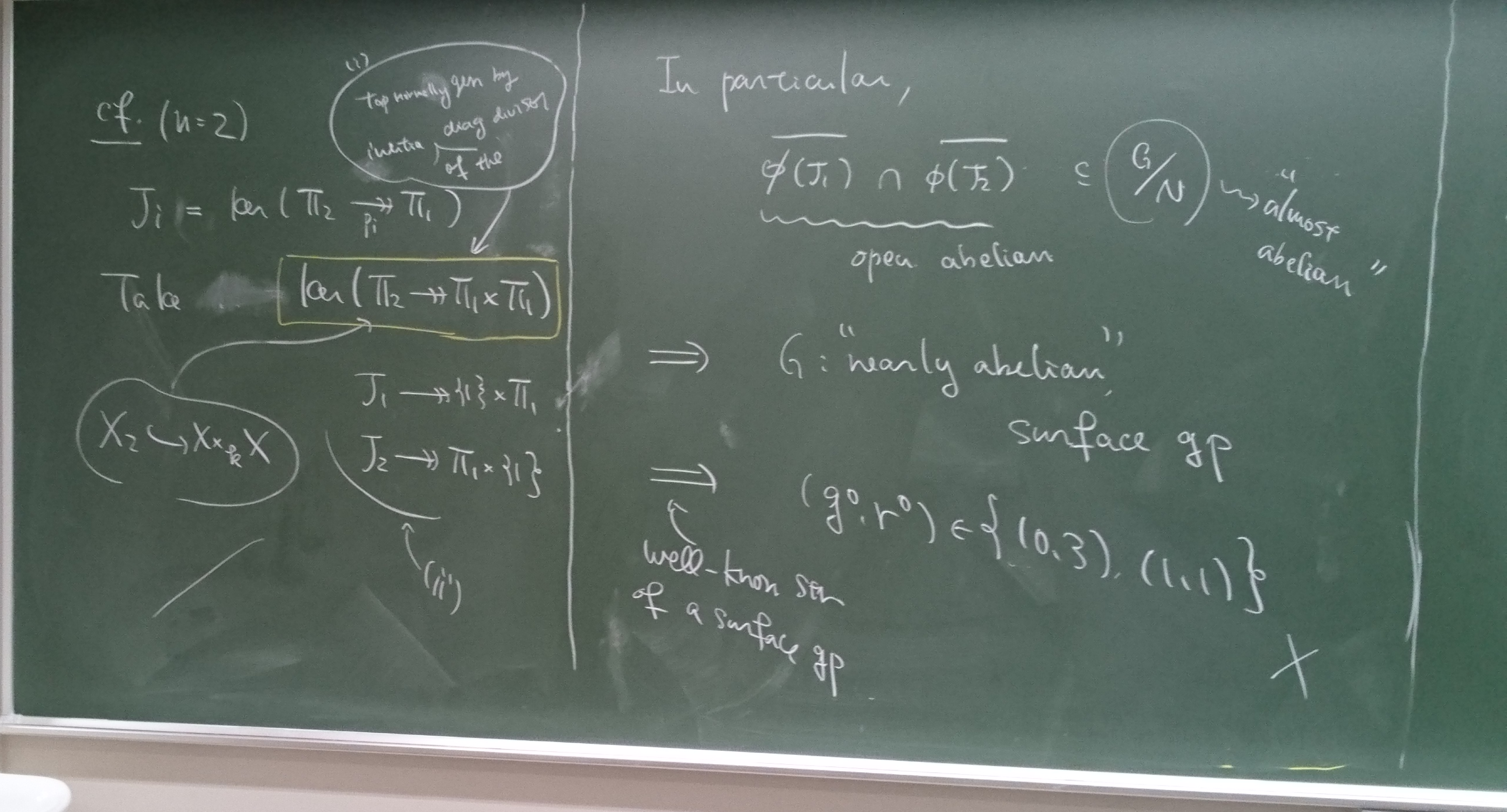
Since G is elastic, it follows that

$\phi(J_1).\phi(J_2)$: open $\subseteq G$

Note : $^{\exists}N \subseteq G$ closed normal s.t.

(i) top normally gen by a single element

(ii) the images of $\phi(J_1).\phi(J_2)$ in G/N commute.



cf. $(n=2)$

(i) Top normally gen by inertia of the diag divisor

$J_i = \ker(\Pi_2 \underset{p_i}{\twoheadrightarrow} \Pi_1)$

Take $\boxed{\ker(\Pi_2 \twoheadrightarrow \Pi_1 \times \Pi_1)}$

$X_2 \hookrightarrow X \times_k X$

$J_1 \twoheadrightarrow \{1\} \times \Pi_1$

$J_2 \twoheadrightarrow \Pi_1 \times \{1\}$

(ii)

In particular,

$$\underbrace{\overline{\phi(J_1)} \cap \overline{\phi(J_2)}}_{\text{open abelian}} \subseteq G/N \rightsquigarrow \text{"almost abelian"}$$

$\Longrightarrow$ G: "nearly abelian" surface gp

$\Longrightarrow$ $(g^0, r^0) \in \{(0,3), (1,1)\}$

well-known str of a surface gp

$N)$ ⇝ "almost abelian"

e gp

$(1,1)\}$

$\times$

In light of claim, we have

$$1 \longrightarrow \alpha(K^{\circ})/K^{\bullet} \longrightarrow \Pi_n^{\bullet}/K^{\bullet} \longrightarrow \Pi_n^{\circ}/K^{\circ} \longrightarrow 1$$

↑ surface gp ⇒ elastic

If $\alpha(K^{\circ})/K^{\bullet} \neq \{1\}$ $(\Rightarrow \text{open})$

then Π_n°/K° : fin gp $\times$

$\therefore \alpha(K^{\circ}) = K^{\bullet}$

$$\Rightarrow \left\{ \begin{array}{l} \text{fib subgp} \subseteq \Pi_n^{\circ} \\ \text{of co-len} = 1 \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{fib subgp} \subseteq \Pi_n^{\bullet} \\ \text{of co-len } 1 \end{array} \right\}$$

$$K^{\circ} \longmapsto K^{\bullet}$$

$\#n$ $\qquad$ $\#n$

⇝ bij //